

CHARACTERISTIC POLYNOMIALS OF COLOR MARKED GRAPHS, RELATED TO THE NORMAL FORM OF THE NON LINEAR SCHRÖDINGER EQUATION

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1. INTRODUCTION

1.1. **Some back ground.** Consider the non linear Schrödinger equation(NLS):

$$(1) \quad iu_t - \Delta u = \kappa |u|^q u + \partial_{\bar{u}} G(|u|^2), q \geq 1 \in \mathbb{N}$$

where $u = u(t, \varphi), \varphi \in \mathbb{T}^n$, $G(a)$ is a real analytic function whose Taylor series start from the degree $q + 2$. One can rescale the constant $\kappa = \pm 1$. Passing to the Fourier representation:

$$(2) \quad u(t, \varphi) = \sum_{k \in \mathbb{Z}^n} u_k(t) e^{(k, \varphi)}; [u]_k := u_k.$$

It is well-known that the equation (1) can be written as an infinite dimensional Hamiltonian dynamical system $\dot{u} = \{H, u\}$, with Hamiltonian:

$$(3) \quad H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{k \in \mathbb{Z}^n: \sum_{i=1}^{2q+2} (-1)^i k_i = 0} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} \dots u_{2q+1} \bar{u}_{2q+2} + [G(|u|^2)]_0$$

on the scale of complex Hilbert spaces:

$$(4) \quad \bar{\ell}^{a,p} := \{u = \{u_k\}_{k \in \mathbb{Z}^n} \mid \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{2a|k|} |k|^{2p} := \|u\|_{a,p}^2 < \infty; a > 0, p > n/2\}$$

For ϵ sufficient small there is a analytic change of variables which brings (3) to $H = H_N + P^{2q+2}(u)$, where $P^{2q+2}(u)$ is analytic of degree at least $2(q + 2)$ in u while:

$$(5) \quad H_N := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\alpha, \beta \in (\mathbb{Z}^n)^{\mathbb{N}}: |\alpha|=|\beta|=q+1; \sum_k (\alpha_k - \beta_k) k = 0, \sum_k (\alpha_k - \beta_k) |k|^2 = 0} \binom{q+1}{\alpha} \binom{q+1}{\beta} u^\alpha \bar{u}^\beta$$

Let us now partition:

$$\mathbb{Z}^n = S \cup S^c; S = \{v_1, \dots, v_m\}$$

where S is called *tangential sites*, it is some (arbitrarily large) subset of \mathbb{Z}^n satisfying the *completeness condition*, S^c -*normal sites*.

We set

$$(6) \quad u_k := z_k, k \in S^c, u_{v_i} := \sqrt{\xi_i + y_i} e^{ix_i} = \sqrt{\xi_i} (1 + \frac{y_i}{2\xi_i} + \dots) e^{ix_i} \text{ for } i = 1, \dots, m,$$

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considering ξ_i as parameters, $|y_i| < \xi_i$, while $y, x, w := z, \bar{z}$ are dynamical variables. We separate $H = N + P$ where N is the *normal* form and collects all the terms of H_N of degree ≤ 2 . We introduce

$$(7) \quad A_r(\xi_1, \dots, \xi_m) = \sum_{\sum_i k_i = r} \binom{r}{k_1, \dots, k_m}^2 \prod_i \xi_i^{k_i}$$

By Proposition 4.4 in [2] we have

$$(8) \quad N = (\omega(\xi), y) + \sum_k \Omega_k(\xi) |z_k|^2 + Q_M(x, \omega)$$

where

$$(9) \quad \omega = \omega_0 + \nabla_\xi A_{q+1}(\xi), \Omega_k = |k|^2 + (q+1)^2 A_{q+1}(\xi)$$

and Q_M is given by formula 12. Let $\{e_1, \dots, e_m\}$ be a basis of \mathbb{Z}^m .

Definition 1.1. (*edges*) Consider the elements:

$$(10) \quad X_q := \left\{ \ell = \sum_{j=1}^{2q} \pm e_{ij} = \sum_{i=1}^m \ell_i e_i, \ell \neq 0, -2e_i, \eta(\ell) \in \{0, -2\} \right\}$$

The support of an edge $\ell = \sum_i n_i e_i$ is the set of indices i with $n_i \neq 0$.

We have $\sum_i |\ell_i| \leq 2q$ and have imposed the mass constraint $\sum_i \ell_i = \eta(\ell) \in \{0, -2\}$. We call all the elements respectively the *black*, $\eta(\ell) = 0$ and *red* $\eta(\ell) = -2$ edges and denote them by X_q^0, X_q^{-2} respectively. Notice that by our constraints the support of an edge contains at least 2 elements.

Definition 1.2. • When $\ell \in X_q^0$, we define \mathcal{P}_ℓ as the set of pairs k, h satisfying $\sum_{j=1}^m \ell_j v_j + k - h = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |k|^2 - |h|^2 = 0$.
• When $\ell \in X_q^{-2}$, we define \mathcal{P}_ℓ as the set of unordered pairs $\{h, k\}$ satisfying $\sum_{j=1}^m \ell_j v_j + k + h = 0; \sum_{j=1}^m \ell_j |v_j|^2 + |k|^2 + |h|^2 = 0$.

For every edge ℓ , set $\ell = \ell^+ - \ell^-$ and define

$$(11) \quad c(\ell) = c_q(\ell) := \begin{cases} (q+1)^2 \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q} \binom{q}{\ell^+ + \alpha} \binom{q}{\ell^- + \alpha} \xi^\alpha, & \ell \in X_q^0; \\ (q+1)q \xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q-1} \binom{q+1}{\ell^- + \alpha} \binom{q-1}{\ell^+ + \alpha} \xi^\alpha, & \ell \in X_q^{-2}. \end{cases}$$

Then in [2] it has been proved that:

$$(12) \quad Q_M(x, \omega) = \sum_{\ell \in X_q^0} c(\ell) e^{\ell x} \sum_{(h, k) \in \mathcal{P}_\ell} z_h \bar{z}_k + \sum_{\ell \in X_q^{-2}} c(\ell) \sum_{h, k \in \mathcal{P}_\ell} (e^{\ell x} z_h z_k + e^{-\ell x} \bar{z}_h \bar{z}_k)$$

This is a very complicated infinite dimensional quadratic Hamiltonian, one needs to decompose this infinite dimensional system into infinitely many decoupled finite dimensional systems.

Definition 1.3. Momentum is the linear map $\pi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n, \pi(e_i) = v_i$.

Define $iM(x)$ as the matrix of $ad(Q_M)$ in the basis z_k, \bar{z}_k , while iM is the matrix of $ad(Q_M)$ in the basis $e^{i\mu x} z_k, e^{-i\mu x} \bar{z}_k, \pi(\mu) + k = 0$. In [2] it was proved that M is block diagonal with 2 blocks (denoted by A, \pm in correspondence with each connected component A of the geometric graph (c. f. 1.2)). The control of these blocks is then needed to prove further non-degeneracy properties of this Hamiltonian. Set

$$\mathbb{Z}_c^m = \{\mu \in \mathbb{Z}^m \mid -\pi(\mu) \in S^c\}.$$

Definition 1.4. The graph $\tilde{\Gamma}_S$ has as vertices the variables z_h, \bar{z}_k and edges corresponding to the non-zero entries of matrix $M(x)$ in the geometric basis (i. e. z_h, \bar{z}_k). The graph Λ has as vertices $\mathbb{Z}_c^m \times \mathbb{Z}/(2)$ and edges corresponding to the non-zero entries of M in the frequency basis (i. e. $e^{i\mu x} z_k, e^{-i\mu x} \bar{z}_k, \pi(\mu) + k = 0$).

Definition 1.5. Two points $h, k \in S^c$ are connected by a black edge if z_h, z_k are connected in $\tilde{\Gamma}_S$, while $h, k \in S^c$ are connected by a red edge if z_h, \bar{z}_k are connected in $\tilde{\Gamma}_S$.

Take a connected component A of Γ_S . Consider block $M_{A,+}$. Given two elements $a \neq b \in A$. By formula 12 the matrix element $M_{a,b}$ is non-zero if and only if they are joined by an edge ℓ and then $M_{a,b} = c(\ell)$ if $b = e^{i\mu x} z_k$ or $M_{a,b} = -c(\ell)$ if $b = e^{-i\mu x} \bar{z}_k$.

$$M_{A,-} = -\bar{M}_{A,+}$$

In order to describe the matrix iN_A of $ad(N)$ on A , we have to finally compute the diagonal terms. One contribution comes from 9 and assumes the value $\nabla_\xi A_{q+1}(\xi) \cdot \mu$ on the element $e^{i\mu x} z_k$.

In application of the KAM algorithm to our Hamiltonian a main point is to prove the validity of the second Melnikov condition. The problem arises in the study of the second Melnikov equation where we have to understand when it is that two eigenvalues are equal or opposite. The condition for a polynomial to have distinct roots is the non-vanishing of the discriminant while the condition for two polynomials to have a root in common is the vanishing of the resultant. In our case these resultants and discriminants are polynomials in the parameters ξ_i so, in order to make sure that the singularities are only in measure 0 sets (in our case even an algebraic hypersurface), it is necessary to show that these polynomials are formally non-zero. This is a purely algebraic problem involving, in each dimension n , only finitely many explicit polynomials and so it can be checked by a finite algorithm. The problem is that, even in dimension 3, the total number of these polynomials is quite high (in the order of the hundreds or thousands) so that the algorithm becomes quickly non practical. In order to avoid this we have experimented with a conjecture which is stronger than the mere non-vanishing of the desired polynomials. We expect our polynomials to be irreducible and separated, in the sense that the connected component of the graph giving rise to the block and its polynomial can be recovered from the associated characteristic polynomial.

1.2. A geometric graph. To the set S we associate the following configuration, given two distinct elements $v_i, v_j \in S$ construct the sphere $S_{i,j}$ having the two vectors as opposite points of a diameter and the two hyperplanes, $H_{i,j}, H_{j,i}$, passing through v_i and v_j respectively, and perpendicular to the line though the two vectors v_i, v_j .

From this configuration of spheres and pairs of parallel hyperplanes we deduce a combinatorial colored graph, denoted by Γ_S , with vertices the points in \mathbb{R}^n and two types of edges, which we call *black* and *red*.

- A black edge connects two points $p \in H_{i,j}$, $q \in H_{j,i}$, such that the line p, q is orthogonal to the two hyperplanes, or in other words $q = p + v_j - v_i$.
- A red edge connects two points $p, q \in S_{i,j}$ which are opposite points of a diameter.

The Problem The problem consists in the study of the connected components of this graph. Of course the nature of the graph depends upon the choice of S but one expects a relatively simple behavior for S generic. It is immediate by the definitions that the points in S are all pairwise connected by black and red edges and it is not hard to see that, for generic values of S , the set S is itself a connected component which we call the special component.

1.3. The Cayley graphs. In order to understand the graph Γ_S we develop a formal setting. Let G be a group and $X = X^{-1} \subset G$ a subset.

1.3.1. Marked graphs.

Definition 1.6. An X -marked graph is an oriented graph Γ such that each oriented edge is marked with an element $x \in X$.

$$a \xrightarrow{x} b \qquad a \xleftarrow{x^{-1}} b$$

We mark the same edge, with opposite orientation, with x^{-1} . Notice that if $x^2 = 1$ we may drop the orientation of the edge.

1.3.2. Cayley graphs. A typical way to construct an X -marked graph is the following. Consider an action $G \times A \rightarrow A$ of G on a set A , we then define.

Definition 1.7 (Cayley graph). The graph A_X has as vertices the elements of A and, given $a, b \in A$ we join them by an oriented edge $a \xrightarrow{x} b$, marked x , if $b = xa$, $x \in X$.

A special case is obtained when G acts on itself by left (resp. right) multiplication and we have the Cayley graph G_X^l (resp. G_X^r). We concentrate on G_X^l which we just denote by G_X .

1.3.3. The linear rules. Denote by $\mathbb{Z}^m := \{\sum_{i=1}^m a_i e_i, a_i \in \mathbb{Z}\}$ the lattice with basis the elements e_i . We consider the group $G := \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ semi-direct product. Its elements are couples (a, σ) with $a \in \mathbb{Z}^m$, $\sigma = \pm 1$. It will be notationally convenient to identify by a the element $(a, +1)$ and by τ the element $(0, -1)$. Note the commutation rules $a\tau = \tau(-a)$. Sometimes we refer to the elements $a = (a, +1)$ as *black* and $a\tau = (a, -1)$ as *red*.

Definition 1.8. We set Λ to be the Cayley graph associated to the elements $X_q := X_q^0 \cup X_q^{-2}$.

1.3.4. From the combinatorial to the geometric graph. In our geometric setting, we have chosen a list S of vectors v_i and we then define $\pi : \mathbb{Z}^m \rightarrow \mathbb{R}^n$ by $\pi : e_i \mapsto v_i$.

We then think of G also as linear operators on \mathbb{R}^n by setting

$$(13) \qquad ak := -\pi(a) + k, \quad k \in \mathbb{R}^n, \quad a \in \mathbb{Z}^m, \quad \tau k = -k$$

We extend $\pi : \mathbb{Z}^m \rightarrow \mathbb{R}^n$ to $\mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ by setting $\pi(a\tau) := \pi(a)$ so that $-\pi$ is just the orbit map of 0 associated to the action (13) (the sign convention is suggested by the conservation of momentum in the NLS).

We then have that X defines also a Cayley graph on \mathbb{R}^n and in fact the graph Γ_S is a subgraph of this graph.

There are symmetries in the graph. The symmetric group S_m of the $m!$ permutations of the elements e_i preserves the graph. We have the right actions of G , on the graph: The sign change

$$(14) \quad (b, \sigma) \mapsto (b, \sigma)\tau = b\sigma\tau, \quad (b, \sigma) \mapsto (b, \sigma)a = (b + \sigma a, \sigma), \quad \forall a, b \in \mathbb{Z}^m.$$

Up to the G action any subgraph can be translated to one containing 0.

We give definitions which are useful to describe the graphs that appear in our construction.

Definition 1.9. A complete marked graph, on a set $A \subset \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ is the full sub-graph generated by the vertices in A .

Definition 1.10. A graph A with $k + 1$ vertices is said to be of dimension k

1.4. Characteristic polynomials of complete color marked graphs. As we said in 1.1 for every complete color marked graph \mathcal{G} we will consider the matrix $C_{\mathcal{G}}$ indexing by vertices of \mathcal{G} as computed in [2] §11. 1. 1:

Given $(a, \sigma), a = \sum_{i=1}^m n_i e_i$ set

$$(15) \quad (q + 1)a(\xi) := \sum_{i=1}^m n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi)$$

then

- In the diagonal at the position $(a, \sigma), a = \sum_{i=1}^m n_i e_i$ we put

$$(16) \quad \begin{cases} (q + 1)a(\xi) & \text{if } \sigma = 1 \\ -(q + 1)a(\xi) + 2(q + 1)^2 A_q(\xi) & \text{if } \sigma = -1 \end{cases}$$

- At the position $((a, \sigma_a), (b, \sigma_b))$ we put 0 if they are not connected, otherwise we put $\sigma_b c(\ell)$ (c. f. 11, where ℓ is the edge connecting a, b).

Define $\chi_{\mathcal{G}} = \chi_{C_{\mathcal{G}}}(t) = \det(tI - C_{\mathcal{G}})$ - the characteristic polynomial of $C_{\mathcal{G}}$.

As we said in 1.1 in order to check the second Melnikov condition we expect that $\chi_{\mathcal{G}}$ are irreducible over \mathbb{Z} and separated. In [3] we have proved this for the case $q = 1, n \in \mathbb{N}$. Here we start to prove irreducibility and separation for bigger q and low dimensions.

2. IRREDUCIBILITY OF CHARACTERISTIC POLYNOMIALS

Lemma 2.1. For any $a \in \mathbb{Z}^m$: $a(\xi)$ has integer coefficients.

Proof. Let $a = \sum_i n_i e_i$. We have

$$\begin{aligned} \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) &= \sum_{\beta \in \mathbb{N}^m; |\beta|_1 = q+1; \beta_i \geq 1} \binom{q+1}{\beta}^2 \beta_i \xi_1^{\beta_1} \dots \xi_i^{\beta_i-1} \dots \xi_m^{\beta_m} \\ \binom{q+1}{\beta}^2 \beta_i &= \binom{q+1}{\beta} \binom{q}{\beta_1, \dots, \beta_i-1, \dots, \beta_m} (q+1) \end{aligned}$$

is divisible by $q + 1$. □

Hence all diagonal elements of $C_{\mathcal{G}}$ are divisible by $q + 1$. Besides by the formula 11 all off-diagonal elements of $C_{\mathcal{G}}$ are also divisible by $q + 1$. Thus we can write:

$$C_{\mathcal{G}} = (q + 1)\tilde{C}_{\mathcal{G}} \Rightarrow \chi_{C_{\mathcal{G}}}(t) = \det(tI - C_{\mathcal{G}}) = \det((q + 1)\tilde{t}I - (q + 1)\tilde{C}_{\mathcal{G}}) = (q + 1)^{n+1} \chi_{\tilde{C}_{\mathcal{G}}}(\tilde{t})$$

So in order to prove the irreducibility and the separation of the polynomials χ_{C_G} it is enough to prove the irreducibility and the separation of the polynomials $\chi_{\tilde{C}_G}$. For simplicity we will denote $\chi_{\tilde{C}_G}$ also by χ_G , and we will redefine $c(\ell)$ by division the right hand sides of 11 by $q + 1$:

$$(17) \quad c(\ell) = c_q(\ell) := \begin{cases} (q+1)\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q} \binom{q}{\ell^+ + \alpha} \binom{q}{\ell^- + \alpha} \xi^\alpha, & \ell \in X_q^0; \\ q\xi^{\frac{\ell^+ + \ell^-}{2}} \sum_{\alpha \in \mathbb{N}^m; |\alpha + \ell^+|_1 = q-1} \binom{q+1}{\ell^+ + \alpha} \binom{q-1}{\ell^- + \alpha} \xi^\alpha, & \ell \in X_q^{-2}. \end{cases}$$

Take a complete colored marked graph \mathcal{A} and compute its characteristic polynomial $\chi_{\mathcal{A}}(t)$. We have:

Theorem 2.1. *When we set a variable $\xi_i = 0$ in $\chi_{\mathcal{A}}(t)$ we obtain the product of the polynomials $\chi_{\mathcal{A}_i}(t)$ where the \mathcal{A}_i are the connected components of the graph obtained from \mathcal{A} by deleting all the edges in which i appears as index, with the induced markings (with $\xi_i = 0$).*

Proof. This is immediate from the form of the matrices. \square

Remark 2.1.

$$(18) \quad \frac{\partial}{\partial \xi_i} A_{q+1}(\xi)|_{\xi_i = \xi_j} = \frac{\partial}{\partial \xi_j} A_{q+1}(\xi)|_{\xi_i = \xi_j} \forall i, j$$

Remark 2.2. *Let $b = \sum_{i=1}^k n_i e_i, n_i \neq 0; \sum_{i=1}^k n_i = 0$. Then:*

$$(19) \quad b(\xi)|_{\xi_1 = \xi_2 = \dots = \xi_k} = 0$$

Proof. By the remark 2.1 we have:

$$b(\xi)|_{\xi_1 = \xi_2 = \dots = \xi_k} = \sum_{i=1}^k n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi)|_{\xi_1 = \xi_2 = \dots = \xi_k} = \frac{\partial}{\partial \xi_1} A_{q+1}(\xi)|_{\xi_1 = \xi_2 = \dots = \xi_k} \sum_{i=1}^k n_i = 0$$

.

\square

Remark 2.3. *Let $\ell = \ell^+ - \ell^-$ be an edge. We have:*

- i) *If ℓ is a black edge, then $|\ell^+|_1 = |\ell^-|_1 \leq q$.*
- ii) *If ℓ is a red edge, then $|\ell^+|_1 \leq q-1, |\ell^-|_1 \leq q+1$.*

Proof. By the definition of edges we have :

$$(20) \quad |\ell^+|_1 + |\ell^-|_1 \leq 2q.$$

On the other hand:

- i) If ℓ is a black edge, then

$$(21) \quad |\ell^+|_1 - |\ell^-|_1 = 0.$$

From (20) and (21) we get $|\ell^+|_1 = |\ell^-|_1 \leq q$.

- ii) If ℓ is red edge, then

$$(22) \quad |\ell^+|_1 - |\ell^-|_1 = -2.$$

From (20) and (22) we get $|\ell^+|_1 \leq q-1, |\ell^-|_1 \leq q+1$. \square

Remark 2.4. : Let $\ell = \sum_{i=1}^k n_i e_i = \ell^+ - \ell^-$, $n_i \neq 0$, be an edge.

- i) If ℓ is a black edge and $k = m$, then $|\ell^+|_1 = |\ell^-|_1 = q$ and $c(\ell) = (q+1)\xi^{(\ell^+ + \ell^-)/2} \binom{q}{\ell^+} \binom{q}{\ell^-}$.
- ii) If ℓ is a red edge and $k = m$, then $|\ell^+|_1 = q - 1, |\ell^-|_1 = q + 1$ and $c(\ell) = q\xi^{(\ell^+ + \ell^-)/2} \binom{q+1}{\ell^-} \binom{q-1}{\ell^+}$.

Proof. Since $S = \{v_1, \dots, v_m\}$ is some arbitrarily large set, we may suppose $m \geq 2q$. If $k = m$ then $|\ell^+|_1 + |\ell^-|_1 = \sum_{i=1}^m n_i \geq m \geq 2q$. Moreover, by definition of edges $\sum_{i=1}^m n_i \leq 2q$. Hence:

$$(23) \quad |\ell^+|_1 + |\ell^-|_1 = \sum_{i=1}^m n_i = 2q.$$

i) When ℓ is a black edge, we have

$$(24) \quad |\ell^+|_1 - |\ell^-|_1 = 0$$

From (23) and (24) we get $|\ell^+|_1 = |\ell^-|_1 = q$. By formula (17) we obtain $c(\ell) = (q+1)\xi^{(\ell^+ + \ell^-)/2} \binom{q}{\ell^+} \binom{q}{\ell^-}$.

ii) When ℓ is a red edge, we have

$$(25) \quad |\ell^+|_1 - |\ell^-|_1 = -2$$

From (23) and (25) we get $|\ell^+|_1 = q - 1, |\ell^-|_1 = q + 1$. By formula (17) we obtain $c(\ell) = q\xi^{(\ell^+ + \ell^-)/2} \binom{q+1}{\ell^-} \binom{q-1}{\ell^+}$. \square

We finally recall Proposition 14 of [2]

Proposition 1. (i) For $n = 1$ and for generic choices of S , all the connected components of Γ_S are either vertices or single edges.

(ii) For $n = 2$, and for every m there exist infinitely many choices of generic tangential sites $S = \{v_1, \dots, v_m\}$ such that, if A is a connect component of the geometric graph Γ_S , then A is either a vertex or a single edge.

Obtained results: For graphs reduced to one vertex the statement is trivial. At the moment we are able to prove the irreducibility and separation in dimension 1, and dimension 2, under the assumptions of Proposition 1 for all q since all graphs which appear have at most one edge.

2.1. One edge.

Theorem 2.2. For any q and any connected color marked graph with one edge the characteristic polynomial is irreducible.

Proof. We choose the root so that the graph has one of the forms:

$$0 \xrightarrow[\text{black}]{\ell} \ell \quad \text{or} \quad 0 \xrightarrow[\text{red}]{\ell} \ell$$

Let $\ell = \sum_{i=1}^k n_i e_i$, $n_i \neq 0$. We have

$$(26) \quad \ell(\xi) = \frac{1}{q+1} \sum_{i=1}^k n_i \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) = \sum_{i=1}^k n_i \sum_{\beta \in \mathbb{N}^m; |\beta|_1 = q+1; \beta_i \geq 1} \binom{q+1}{\beta} \binom{q}{\beta_1, \dots, \beta_i - 1, \dots, \beta_m} \xi_1^{\beta_1} \dots \xi_i^{\beta_i - 1} \dots \xi_m^{\beta_m}$$

Set $\bar{\ell}(\xi) := \ell(\xi)$ if $\eta(\ell) = 0$ and $\bar{\ell}(\xi) := -\ell(\xi) + 2(q+1)A_q(\xi)$ if $\eta(\ell) = -2$.

Remark 2.5. For every i in the support of ℓ , unless $q = 4$ and $\ell = -5e_i + e_j + e_k + e_m$, the polynomial $\bar{\ell}(\xi)$ contains the term ξ_i^q with non zero coefficient.

Proof: In the formula of $\ell(\xi)$ there is the monomial:

$$(n_i + (q+1) \sum_{h \neq i} n_h) \xi_i^q,$$

since $\sum_h n_h = \eta(\ell)$ this equals

$$-qn_i \xi_i^q \quad \text{if } \eta(\ell) = 0$$

and

$$[n_i + (q+1)(-2 - n_i)] \xi_i^q \quad \text{if } \eta(\ell) = -2$$

In $A_q(\xi)$ the monomial ξ_i^q appears with coefficient 1, so we get in $\bar{\ell}$ the coefficient of ξ_i^q is:

$$(27) \quad -n_i + (q+1)(2 + n_i) + 2(q+1) = 4(q+1) + qn_i$$

which is non zero unless $q = 4, n_i = -5$ or $q = 2, n_i = -6$; $q = 1, n_i = -4$, by Formula (10) these last two cases do not occur since $|n_i| \leq 2q$. As for $q = 4$ an edge is a sum of 8 elements e_i with sign ± 1 , if there is a coefficient -5 at least 5 elements have coefficient -1 , then there must be 3 elements with coefficient 1 to give $\eta(\ell) = -2$. This extra case will be considered at the end of this subsection. \square

We now compute with the matrix

$$C_{\mathcal{G}} = \begin{pmatrix} 0 & \sigma_{\ell} c(\ell) \\ c(\ell) & \bar{\ell}(\xi) \end{pmatrix}$$

$$(28) \quad \chi_{\mathcal{G}}(t) = \det \begin{pmatrix} t & -\sigma_{\ell} c(\ell) \\ -c(\ell) & t - \bar{\ell}(\xi) \end{pmatrix} = t^2 - \bar{\ell}(\xi)t - \sigma_{\ell} c(\ell)^2.$$

Suppose that $\chi_{\mathcal{G}}$ is not irreducible, then:

$$(29) \quad \chi_{\mathcal{G}}(t) = (t + r(\xi))(t - \bar{\ell}(\xi) - r(\xi)).$$

Compare the free coefficients in 28 and 29 we get

$$(30) \quad r(\xi)(-\bar{\ell}(\xi) - r(\xi)) = -\sigma_{\ell} c(\ell)^2.$$

By the formula 11 $c(\ell)^2$ is divisible by $\xi_i^{|n_i|}, \forall i = 1, \dots, k$.

For any i if $r(\xi)$ is divisible by ξ_i , by remark 2.5 $\bar{\ell}(\xi)$ is not divisible by ξ_i , then $-\bar{\ell}(\xi) - r(\xi)$ is not divisible by ξ_i . And inversely, if $-\bar{\ell}(\xi) - r(\xi)$ is divisible by ξ_i , then $r(\xi)$ is not divisible by ξ_i . Hence we have:

$$(31) \quad r(\xi) = \xi_i^{|n_i|} s_i, i \in A$$

$$(32) \quad -\bar{\ell}(\xi) - r(\xi) = \xi_j^{|n_j|} u_j, j \in B.$$

where $A \cup B = \{1, \dots, k\}; A \cap B = \emptyset$.

(1) If $A \neq \emptyset$ and $B \neq \emptyset$, then for some couple i, j we have:

$$(33) \quad \bar{\ell}(\xi) = -(\xi_i^{|n_i|} s_i + \xi_j^{|n_j|} u_j)$$

From remark 2.5 we must have $n_h = 0, \forall h \neq i, j$,

(a) **When ℓ is a black edge:**

We have $\sigma_\ell = 1$ and by the definition of edge (cf. 1.1) $\ell = ne_i - ne_j; 2|n| \leq 2q$.

We may suppose $i = 1, j = 2, n > 0$. We have $\bar{\ell}(\xi) = \ell(\xi)$ and:

$$(34) \quad \ell(\xi) = n \left(\sum_{\beta \in \mathbb{N}^m; |\beta|_1 = q+1, \beta_1 \geq 1} \binom{q+1}{\beta} (\beta_1 - 1, \beta_2, \dots, \beta_m)^q \xi_1^{\beta_1-1} \xi_2^{\beta_2} \dots \xi_m^{\beta_m} - \right. \\ \left. - \sum_{\beta' \in \mathbb{N}^m; |\beta'|_1 = q+1; \beta'_2 \geq 1} \binom{q+1}{\beta'} (\beta'_1, \beta'_2 - 1, \dots, \beta'_m)^q \xi_1^{\beta'_1} \xi_2^{\beta'_2-1} \dots \xi_m^{\beta'_m} \right)$$

Remark that

$$\xi_1^{\beta_1-1} \xi_2^{\beta_2} \dots \xi_m^{\beta_m} = \xi_1^{\beta'_1} \xi_2^{\beta'_2-1} \dots \xi_m^{\beta'_m} \Leftrightarrow \beta_1 - 1 = \beta'_1, \beta_2 = \beta'_2 - 1, \beta_i = \beta'_i \forall i \geq 3$$

Then:

$$(35) \quad \ell(\xi) = n \sum_{\beta \in \mathbb{N}^m, |\beta|_1 = q+1, \beta_1 \geq 1} \frac{q!}{(\beta_1 - 1)! \beta_2! \dots \beta_m!} \frac{(q+1)!}{\beta_1! \dots \beta_m!} \left(1 - \frac{\beta_1}{\beta_2 + 1}\right) \xi_1^{\beta_1-1} \xi_2^{\beta_2} \dots \xi_m^{\beta_m}$$

By 33 we must have

$$(36) \quad \ell(\xi) = -(\xi_1^n s_1 + \xi_2^n u_2).$$

(i) If $n > 1$, we take $\beta_1 = 1, \beta_2 = n - 1, \beta_3 = q + 1 - n, \beta_4 = \dots = \beta_m = 0$, then in the formula 35 of $\ell(\xi)$, there is the monomial

$$n \frac{q!}{(n-1)!(q+1-n)!} \frac{(q+1)!}{(n-1)!(q+1-n)!} \left(1 - \frac{1}{n}\right) \xi_2^{n-1} \xi_3^{q+1-n} \neq 0$$

and they are not divisible by ξ_1^n or ξ_2^n . This contradicts 36.

(ii) $n = 1$. We have $\ell^+ = (1, 0, \dots, 0); \ell^- = (0, 1, \dots, 0)$. Then from 17 we get

$$(37) \quad c(\ell)^2 = (q+1)^2 \xi_1 \xi_2 \left(\sum_{\alpha \in \mathbb{N}^m: \sum_i \alpha_i + 1 = q} \binom{q}{\alpha_1 + 1, \alpha_2, \dots, \alpha_m} \binom{q}{\alpha_1, \alpha_2 + 1, \dots, \alpha_m} \right) \xi^\alpha)^2$$

Let p be a prime divisor of $q+1: q+1 = p^k u, g.c.d(p, u) = 1$. We have:

$$(38) \quad \chi_{\mathcal{G}} = t(t - \ell(\xi)) \pmod{p} \Rightarrow \chi_{\mathcal{G}} = (t + ps)(t - ps - \ell(\xi))$$

By 28 and 37 the free coefficient of $\chi_{\mathcal{G}}$ must be divisible by p^{2k} :

$$(39) \quad p^{2k} | ps(-\ell(\xi) - ps)$$

By formula (35) we see that the coefficient of the term ξ_1^q is $-q$, the coefficient of the term ξ_2^q is q . One deduces that $\ell(\xi)$ is not divisible by p since $g.c.d(q, q+1) = 1$. Hence $(-\ell(\xi) - ps)$ is not divisible by p . So by 39 we must have $p^{2k-1} | s$. Now take $\xi_1 = \xi_2 \Rightarrow \ell(\xi) = 0$, then the free coefficient of $\chi_{\mathcal{G}}$ when $\xi_1 = \xi_2$ is divisible by p^{4k} . But in 28 when $\xi_1 = \xi_2$ the free coefficient of $\chi_{\mathcal{G}}$ is $-c(\ell)^2|_{\xi_1=\xi_2}$, it is not divisible by p^{4k} , since in 37 if we take $\alpha_1 = \alpha_2 = 0, \alpha_3 = q - 1$, we have the monomial:

$$(q+1)^2 \xi_1^2 (q^2 \xi_3^{q-1})^2$$

is not divisible by p^{4k} .

(b) When ℓ is a red edge: When $h \neq i, j, n_h = 0$ we get the coefficient of the term ξ_h^q in $\bar{\ell}(\xi)$ is $4(q+1) + qn_h = 4(q+1) \neq 0$, so 33 cannot hold.

(2) If $B = \emptyset$, then $A = \{1, \dots, k\}$

$$(40) \quad r(\xi) = \xi_1^{|n_1|} \dots \xi_k^{|n_k|} s$$

(a) When ℓ is a black edge: Take $\xi_1 = \dots = \xi_k$, by the remark we have $\ell(\xi)|_{\xi_1=\dots=\xi_k} = 0$, hence

$$(41) \quad \chi_G(t)|_{\xi_1=\dots=\xi_k} = (t + r(\xi)|_{\xi_1=\dots=\xi_k})(t - r(\xi)|_{\xi_1=\dots=\xi_k}) = t^2 - r(\xi)|_{\xi_1=\dots=\xi_k}^2.$$

By 41 the free coefficient of $\chi_G|_{\xi_1=\dots=\xi_k}$ is divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$. But by 28 the free coefficient of $\chi_G|_{\xi_1=\dots=\xi_k}$ is $-c(\ell)^2|_{\xi_1=\dots=\xi_k}$.

-If $k = m$, then by remark 2.4 $-c(\ell)^2|_{\xi_1=\dots=\xi_k} = -(q+1)^2 \xi_1^{2\sum_{i=1}^k |n_i|} \left(\frac{q}{\ell^+}\right)^2 \left(\frac{q}{\ell^-}\right)^2$

is not divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$.

-If $k < m$, then

$$(42) \quad -c(\ell)^2|_{\xi_1=\dots=\xi_k} = -(q+1)^2 \xi_1^{2\sum_{i=1}^k |n_i|} \left(\sum_{\alpha \in \mathbb{N}^m: |\ell^+ + \alpha|_1 = q} \left(\frac{q}{\ell^+ + \alpha}\right) \left(\frac{q}{\ell^- + \alpha}\right) \xi_1^{\sum_{i=1}^k \alpha_i} \xi_{k+1}^{\alpha_{k+1}} \dots \xi_m^{\alpha_m} \right)^2$$

Take $\alpha_1 = \dots = \alpha_k = 0, \alpha_{k+1} = q - |\ell^+|_1$, we see that $-c(\ell)^2|_{\xi_1=\dots=\xi_k}$ contains

the term $\xi_1^{2\sum_{i=1}^k |n_i|} \xi_{k+1}^{2(q-|\ell^+|_1)}$ with the coefficient $-(q+1)^2 \left(\frac{q}{\ell^+ + \alpha}\right)^2 \left(\frac{q}{\ell^- + \alpha}\right)^2$.

Hence $-c(\ell)^2|_{\xi_1=\dots=\xi_k}$ is not divisible by $\xi_1^{2\sum_{i=1}^k |n_i|}$.

(b) When ℓ is a red edge: Take $\xi_1 = \dots = \xi_k$, we have

$$(43) \quad \frac{\partial}{\partial \xi_i} A_{q+1}(\xi) = \frac{\partial}{\partial \xi_j} A_{q+1}(\xi) \forall i, j \Rightarrow \ell(\xi)|_{\xi_1=\dots=\xi_k} = \sum_{i=1}^k n_i \frac{\partial}{\partial \xi_1} A_{q+1}(\xi) = -2 \frac{\partial}{\partial \xi_1} A_{q+1}(\xi) = -2 \sum_{|\alpha|_1=q+1, \alpha_1 \geq 1} \frac{1}{q+1} \left(\frac{q+1}{\alpha}\right)^2 \alpha_1 \xi_1^{\alpha_1+\alpha_2+\dots+\alpha_k-1} \xi_{k+1}^{\alpha_{k+1}} \dots \xi_m^{\alpha_m}$$

$$(44) \quad A_q(\xi)|_{\xi_1=\dots=\xi_k} = \sum_{\beta: |\beta|_1=q} \left(\frac{q}{\beta}\right)^2 \xi_1^{\beta_1+\dots+\beta_k} \xi_{k+1}^{\beta_{k+1}} \dots \xi_m^{\beta_m}.$$

From (43) and (44) we have

$$(45) \quad -\bar{\ell}(\xi)|_{\xi_1=\dots=\xi_k} = (\ell(\xi) - 2(q+1)A_q(\xi))|_{\xi_1=\dots=\xi_k} = -2 \sum_{\alpha: |\alpha|_1=q+1; \alpha_1 \geq 1} \left(\frac{\alpha_1}{q+1} \left(\frac{q+1}{\alpha}\right)^2 + (q+1) \left(\frac{q}{\alpha_1, \dots, \alpha_m}\right)^2\right) \xi_1^{\alpha_1+\dots+\alpha_k-1} \xi_{k+1}^{\alpha_{k+1}} \dots \xi_m^{\alpha_m} = -2 \sum_{\alpha: |\alpha|_1=q+1; \alpha_1 \geq 1} \left(\frac{\alpha_1}{q+1} \left(\frac{(q+1)!}{\alpha_1! \dots \alpha_m!}\right)^2 - (q+1) \left(\frac{q!}{(\alpha_1-1)! \dots \alpha_m!}\right)^2\right) \xi_1^{\alpha_1+\dots+\alpha_k-1} \xi_{k+1}^{\alpha_{k+1}} \dots \xi_m^{\alpha_m} = 2 \sum_{\alpha: |\alpha|_1=q+1; \alpha_1 > 1} \frac{q!}{(\alpha_1-1)! \dots \alpha_m!} \frac{(q+1)!}{\alpha_1! \dots \alpha_m!} (\alpha_1-1) \xi_1^{\alpha_1+\dots+\alpha_k-1} \xi_{k+1}^{\alpha_{k+1}} \dots \xi_m^{\alpha_m}.$$

Hence $-\ell(\bar{\xi})|_{\xi_1=\dots=\xi_k}$ is divisible by ξ_1 . By (40) $r(\xi)|_{\xi_1=\dots=\xi_k} = \xi_1^{n_1+\dots+n_k}s$ is divisible. Then $(-\ell(\bar{\xi}) - r(\xi))|_{\xi_1=\dots=\xi_k}$ is divisible by ξ_1 . By (30) and (40) we have:

(46)

$$\begin{aligned} \xi_1^{|n_1|} \dots \xi_k^{|n_k|} s(-\bar{\ell}(\xi) - r(\xi)) &= c(\ell)^2 = \xi_1^{|n_1|} \dots \xi_k^{|n_k|} \left(\sum_{\alpha \in \mathbb{N}^m: |\ell^+ + \alpha|_1 = q-1} \binom{q-1}{\ell^+ + \alpha} \binom{q+1}{\ell^- + \alpha} \xi^\alpha \right)^2 \\ \implies s(-\bar{\ell}(\xi) - r(\xi)) &= \left(\sum_{\alpha \in \mathbb{N}^m: |\ell^+ + \alpha|_1 = q-1} \binom{q-1}{\ell^+ + \alpha} \binom{q+1}{\ell^- + \alpha} \xi^\alpha \right)^2 \end{aligned}$$

(47)

$$\implies s(-\bar{\ell}(\xi) - r(\xi)) = \left(\sum_{\alpha \in \mathbb{N}^m: |\ell^+ + \alpha|_1 = q-1} \binom{q-1}{\ell^+ + \alpha} \binom{q+1}{\ell^- + \alpha} \xi^\alpha \right)^2.$$

So the right hand side of (47) when $\xi_1 = \xi_2 = \dots = \xi_k$ must be divisible by ξ_1 . But in fact:

- If $k = m$, then by remark 2.4

$$\left(\sum_{\alpha \in \mathbb{N}^m: |\ell^+ + \alpha|_1 = q-1} \binom{q-1}{\ell^+ + \alpha} \binom{q+1}{\ell^- + \alpha} \xi^\alpha \right)^2 = \binom{q-1}{\ell^+}^2 \binom{q+1}{\ell^-}^2$$

is a constant, not divisible by ξ_1 .

- If $k < m$, take $\tilde{\alpha}$ such that $\tilde{\alpha}_1 = \dots = \tilde{\alpha}_k = 0, \tilde{\alpha}_{k+1} = q-1-\ell^+$ then the right hand side of (47) contains the monomial

$$\binom{q-1}{\ell^+ + \tilde{\alpha}}^2 \binom{q+1}{\ell^- + \tilde{\alpha}}^2 \xi_{k+1}^{2(q-1-|\ell^+|_1)}.$$

Hence the right hand side of (47) is not divisible by ξ_1 .

(3) The case $A = \emptyset, B = \{1, \dots, k\}$ is similar.

Extra case:

$$q = 4, \ell = -5e_i + e_j + e_k + e_m$$

In this case $\sigma_\ell = -1$. By 17 we have:

$$(48) \quad c(\ell) = 4\xi_i^{5/2}\xi_j^{1/2}\xi_k^{1/2}\xi_m^{1/2}.$$

By 30:

$$(49) \quad r(\xi)(-\bar{\ell}(\xi) - r(\xi)) = c(\ell)^2 = 4\xi_i^{5/2}\xi_j^{1/2}\xi_k^{1/2}\xi_m^{1/2}.$$

Hence $r(\xi), -\bar{\ell}(\xi) - r(\xi)$ must be monomials that contains only variables from $\xi_i, \xi_j, \xi_k, \xi_m$. So $\bar{\ell}(\xi)$ must be a polynomial that contains only variables $\xi_i, \xi_j, \xi_k, \xi_m$. But in fact, for h that is not in the support of $\ell, n_h = 0$ and by (27) the coefficient of ξ_h^q in $\bar{\ell}(\xi)$ is $4(q+1) + qn_h = 20$. Hence we have a contradiction. \square

REFERENCES

- [1] C. Procesi, M. Procesi *A normal form of the non-linear Schrödinger equation*, preprint
- [2] C. Procesi, M. Procesi *Normal forms for the non-linear Schrödinger equation*, preprint
- [3] C. Procesi, M. Procesi and Nguyen Bich Van *The energy graph of the nonlinear Schrödinger equation*, preprint.